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Higgs Fields, Curved Space, and Hadron Structure

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Abstract

We show that in completely unified Yang–Mills–Einstein–Higgs-type gauge theories with spontaneous symmetry breaking there exists the possibility that hadrons can be visualized as "microuniverses" where the large curvature within a region of about 0.7×10^{-13}cm arises from a large negative value of the VEV of the Hamiltonian. The low-lying collective excitations of the system have Hooke group symmetry and can be described as multi-quasiparticle systems with oscillator-like energy spectra. The lowest states span reducible representations of $SU(3)$ and correspondence with the naive nonrelativistic quark model can be established. Confinement and absence of nonzero triality excitations can be explained in a natural way.

1. *Introduction*

In the well-explored class of gauge theories with spontaneously broken symmetries the asymmetry of the vacuum is closely related to the nonvanishing vacuum expectation value of a Higgs field. In a canonical formulation this $\langle\phi\rangle$ appears as a constant term in the Lagrangian. If we consider a fully unified gauge theory of the Yang-Mills-Higgs-Einstein type[1] and vary the world Lagrangian with respect to the metric $g_{\mu\nu}$, then the constant $\langle\phi\rangle$ contributes a term of the form $f(V)g_{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$ in the gravitational field equations, where V denotes an effective potential. Clearly, even for empty space, $f(V) \neq 0$. The total energy momentum tensor can then be split into two parts,

$$T_{\mu\nu} = T^V_{\mu\nu} + T^M_{\mu\nu} \tag{1.1}$$

where $T^M_{\mu\nu}$ is the contribution from matter and radiation, and

$$T^V_{\mu\nu} \equiv f(V)g_{\mu\nu} \tag{1.2}$$

[1] See, for example, Cremmer and Sherk (1977), Cho and Freund (1975), Bais and Russell (1975).

is the vacuum contribution. The Einstein equations read

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = -\kappa T^M_{\mu\nu} - \Lambda g_{\mu\nu} \tag{1.3}$$

where we have defined Λ by setting

$$\kappa T^V_{\mu\nu} = \Lambda g_{\mu\nu} \tag{1.4}$$

Thus, the effect of a nonvanishing Higgs field vacuum expectation value manifests itself in the necessary appearence of the cosmological term with

$$\Lambda = \kappa f(V) \tag{1.5}$$

The curvature-modifying effect of spontaneously broken symmetry has been noted recently by several authors[2] and has been persued in various directions, mainly in studying possible cosmological consequences. In this paper we propose to explore the possibility that the solutions of the gravitational field equations (1.3) with Λ given by (1.5) describe strongly curved "microuniverses" with de Sitter-type symmetry which we identify with hadrons. Low-energy excitations of such systems can then be interpreted to correspond to quasiparticles which transform as representations of the Hooke group, the latter being the nonrelativistic (low-speed) limit, in the sense of a speed-space contraction, of the de Sitter group (Bacry and Lévy-Leblond, 1968). Appropriate multiquasiparticle states will be identified with the conventional harmonic oscillator quark model (Faiman and Hendry, 1968), states, where the existence of $SU(3)$ degeneracy within the energy eigenstates of the Hooke system provides the correspondence between quarks and Hooke quasiparticles.

2. *Solution of the Field Equations and Microscopic Effects of Higgs-Mechanism-Induced Curvature*

We seek the simplest class of solutions of the Einstein equations (1.3), i.e., we assume that the line element is spherically symmetric and, with a suitable choice of time coordinate, static. Thus (we shall take $c = 1$ throughout this paper) we have

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\Omega^2 \tag{2.1}$$

where

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta \, d\varphi^2 \tag{2.2}$$

To start with, we take $T^M_{\mu\nu} = 0$; then (1.3) gives[3]

$$e^{\nu} = 1 - \tfrac{1}{3}\Lambda r^2, \qquad e^{\lambda} = (1 - \tfrac{1}{3}\Lambda r^2)^{-1} \tag{2.3}$$

[2] Lind (1974), Veltman (1975), Domokos (1976).

[3] For the subsequent, well-known calculations see, for example, Rindler (1969).

To exhibit the symmetry group of this space, we embed it in a five-dimensional pseudo-Euclidean space. As is well known, the embedding formulas depend on the sign of Λ. For $\Lambda > 0$ we set

$$x_0 = \left(\frac{3}{\Lambda}\right)^{1/2} \left(1 - \frac{\Lambda r^2}{3}\right)^{1/2} \left[\sinh\left(\frac{\Lambda}{3}\right)^{1/2} t\right]$$

$$x_1 = \left(\frac{3}{\Lambda}\right)^{1/2} \left(1 - \frac{\Lambda r^2}{3}\right)^{1/2} \left[\cosh\left(\frac{\Lambda}{3}\right)^{1/2} t\right]$$

$$x_2 = r \sin\theta \cos\phi$$

$$x_3 = r \sin\theta \sin\phi$$

$$x_4 = r \cos\theta$$

which yields the equation of a pseudohyperboloid:

$$x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2 - x_4{}^2 = -\tfrac{1}{3}\Lambda \tag{2.4}$$

On the other hand, if $\Lambda < 0$, we must set

$$x_0 = \left(\frac{3}{|\Lambda|}\right)^{1/2} \left(1 + \frac{|\Lambda|}{3} r^2\right)^{1/2} \left[\sin\left(\frac{|\Lambda|}{3}\right)^{1/2} t\right]$$

$$x_1 = \left(\frac{3}{|\Lambda|}\right)^{1/2} \left(1 + \frac{|\Lambda|}{3} r^2\right)^{1/2} \left[\cos\left(\frac{|\Lambda|}{3}\right)^{1/2} t\right]$$

$$x_2 = r \sinh\theta \cos\phi$$

$$x_3 = r \sinh\theta \sin\phi$$

$$x_4 = r \cosh\theta$$

and obtain the equation of a pseudosphere:

$$x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2 + x_4{}^2 = \tfrac{1}{3}|\Lambda| \tag{2.5}$$

Thus, we see that for either sign of Λ the group of isometries of the solution is of the de Sitter type; for $\Lambda > 0$ the group is isomorphic to $SO(4, 1)$ and for $\Lambda < 0$ it is isomorphic to $SO(3, 2)$.

By a suitable coordinate transformation the line element (2.1) [with (2.2) and (2.3) understood] can be cast into the standard Robertson–Walker form,[3,4] and we have, for $\Lambda > 0$,

$$ds^2 = d\alpha^2 - \frac{3}{\Lambda}\left[\cosh^2\left(\frac{\Lambda}{3}\right)^{1/2} \alpha\right] \; [d^2\chi + (\sin^2\chi) d\Omega^2] \tag{2.6}$$

[4] For definiteness, we assumed positive 3-curvature. For negative (zero) curvature the cosh in front of the appropriately taken bracketed term must be replaced by sinh (exp), respectively.

whereas for $\Lambda < 0$ we obtain

$$ds^2 = d\alpha^2 - \frac{3}{|\Lambda|}\left[\cos^2\left(\frac{|\Lambda|}{3}\right)^{1/2}\alpha\right]\;[d^2\chi + (\sinh^2\chi)d\Omega^2] \qquad (2.7)$$

It is interesting to note that, for $\Lambda < 0$, when the scale factor of the spacelike section of the 3-world at a given time coordinate α is oscillating, the "maximal radius" (amplitude) $A \equiv (3/|\Lambda|)^{1/2}$ happens to be the same as the radius of the 5-hypersphere in the (formal) embedding space, cf. equation (2.5). (A similar remark applies, *mutatis mutandis*, for the case of $\Lambda > 0$.)

In order to study the departure of the empty-space geometry from the flat-space geometry in the case of a spontaneously broken gauge symmetry, we now attempt to express the effective cosmological constant (or equivalently, the "radius" A of the 3-space sections) in terms of the parameters that are characteristic of a spontaneously broken gauge theory. For the sake of orientation, we concentrate on that class of models that contains homogeneous Higgs field terms in the Lagrangian. The Higgs part of the Lagrangian can be written as

$$L_\phi = \partial_\mu\phi^\dagger\partial^\mu\phi + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \qquad (2.8)$$

Introducing new fields ϕ_1 and ϕ_2 via the decomposition $\phi = \phi_1 + i\phi_2$ with ϕ_1, ϕ_2 Hermitian, and allowing one field to acquire a nonvanishing vacuum expectation value σ, we get the formal change

$$L_\phi \to L'_{\phi_1'\phi_2'} = \hat{L}_{\phi_1'\phi_2'} + \tfrac{1}{2}\mu^2\sigma^2 - \tfrac{1}{4}\lambda\sigma^4 \qquad (2.9)$$

where $\hat{L}_{\phi_1'\phi_2'}$ contains all the "operator terms" in the new shifted fields ϕ_1', ϕ_2'. As is well known, stability considerations lead to

$$\sigma^2 = \mu^2/\lambda \qquad (2.10)$$

Because of the presence of nonarbitrary constant terms in the Lagrangian, the vacuum expectation value of the energy density cannot vanish, and from (2.9), using (2.10), we find

$$\langle H\rangle = -\mu^2\sigma^2/4 \qquad (2.11)$$

By reason of covariance, then, $T^V_{\mu\nu} = -(\mu^2\sigma^2/4)g_{\mu\nu}$ so that equation (1.4) yields

$$\Lambda = -\kappa\mu^2\sigma^2/4 \qquad (2.12)$$

Since $\mu^2 > 0$, $\sigma^2 > 0$ we see that Λ is *negative*, so that for our system the oscillating $SO(3, 2)$ de Sitter solution, given above, applies. The effective "radius" or amplitude[5] is given by

$$A = (3/|\Lambda|)^{1/2} = (12/\kappa\mu^2\sigma^2)^{1/2} \qquad (2.13)$$

[5] It is well known (cf. Rindler, 1969) that the oscillating de Sitter solution (sometimes also called "empty Lemaître solution") is an *open* "universe." Thus, while we were led to view hadrons as microuniverses, the "radius" A does not have such a simple interpretation as would be the case for some closed microuniverse. At any rate, A is a characteristic length of the hadron and one should imagine the internal curved space smoothly joined, at A, to the external flat space.

Equation (2.7) also tells us that the circular frequency of the oscillation is

$$\omega = (|\Lambda|/3)^{1/2} = (\kappa\mu^2\sigma^2/12)^{1/2} \tag{2.14}$$

In order to get numerical estimates for Λ, A, and ω, we consider first a Salam-Weinberg-type unified theory of weak and electromagnetic interactions *only*. In that case

$$\sigma^2 = (2^{1/2}G_F)^{-1}, \qquad \mu^2 = m_\phi{}^2/2 \tag{2.15}$$

where G_F is the Fermi weak coupling constant and m_ϕ the Higgs boson mass. Whereas, in the tree approximation, m_ϕ is arbitrary, Weinberg (1976) recently showed that in the one-loop approximation one has the lower bound[6] $m_\phi > 4.91$ GeV. Using this bound and the accepted values for G_F and κ, we obtain from (2.12), (2.13), (2.14), and (2.15) the estimates

$$|\Lambda| > 3 \times 10^4 \text{ cm}^{-2}$$

$$A < 6 \times 10^{-2} \text{ cm}$$

$$\omega > 1 \times 10^{15} \text{ sec}^{-1}$$

Since there is no reason to believe that the actual value of m_ϕ is many orders of magnitude larger than its lower bound, this result does not appear promising, since it predicts a "radius" about 10^{12} times larger than the region where one might reasonably expect significant deviation from flatness. In passing we note that including the effect of the presence of some kind of subnuclear matter on the energy-momentum tensor (i.e., adding to $T^V_{\mu\nu}$ some term $T^M_{\mu\nu}$) has no significant consequence. Indeed, taking this matter to be incoherent dust and maintaining maximal symmetry, the only effect on the solution is to modify everywhere Λ to become $\Lambda + \kappa\rho$, where ρ is the matter density. Adopting the value $\sim 10^{14}$ g cm^{-3}, we find, with the value of Λ calculated above,

$$\kappa\rho/\Lambda \lesssim 10^{-17} \tag{2.16}$$

so that the correction is utterly negligible.

However, a moment's reflection tells us that we are not really in trouble. Indeed, our fundamental premise has been that our underlying world Lagrangian suffices to yield *all* field equations and interactions. It is therefore inconsistent to consider a theory where, besides gravitation, only electronmagnetic and weak interactions are unified: we must visualize possible theories that account for strong interactions as well. It is generally believed that the observed hierarchy of interaction-strength is an effect of the symmetry breaking and that, in particular, the unique superstrong coupling constant determines the scale of the associated Higgs boson masses $m \sim \mu$. If these quantitites are large enough, equation (2.13) tells us that the characteristic "radius" A may become sufficiently small to account for the curved region corresponding to a hadron. For example, in the $SU(5)$-type unified theory of Georgi and Glashow the superheavy bosons must have masses (Georgi et al., 1974) of about 10^{15}-10^{17} GeV. It is also interesting to observe that, in the gauge supersymmetry theory

[6] This holds if one uses for the Weinberg angle the value $\theta_W \approx 35°$. Otherwise, $m_\phi > 3.75$ GeV.

of Arnowitt (1977) (which is the "ultimate" unified gauge theory, since *all* fields are gauge fields), superheavy masses of order as high as 10^{18} GeV appear to be necessary to account for some experimental facts.

In summary, the very real possibility of such superheavy masses and/or large symmetry-breaking effects (reflected in large values of σ) makes our viewpoint viable. Turning the argument around, we observe that we expect

$$A \lesssim 0.7 \times 10^{-13} \text{ cm} \tag{2.17a}$$

(which is the typical length corresponding to a hadron formfactor[7]). Then the relation $A = (3/|\Lambda|)^{1/2}$ gives[8]

$$|\Lambda| \gtrsim 6 \times 10^{26} \text{ cm}^{-2} \tag{2.17b}$$

and from $\omega = (|\Lambda|/3)^{1/2}$ we get

$$\omega \gtrsim 4 \times 10^{23} \text{ sec}^{-1} \tag{2.17c}$$

We note that, since $|\Lambda|$ is so large, effects of subnuclear matter on the curvature are absolutely negligible.

3. *Symmetries and Energy Eigenstates of Hadron Microuniverses*

In order to study the characteristic properties of hadrons conceived as microuniverses, we must first study the symmetry properties and energy excitations of such systems.

Since, as pointed out above, the presence of normal subnuclear matter does not alter the de Sitter-type metric solution of the Einstein equations, the internal excitations of our systems can be considered to be represented by the motion of quasiparticles[9] in the de Sitter world, following essentially free particle trajectories.

We will be interested in the lowest hadronic energy excitations. These low-energy excitations of the quasiparticles correspond to low-velocity motions of the quasiparticles. To see this we observe that, for low-energy excitations the quasiparticles essentially just follow the oscillatory motion of the de Sitter world, and the order of linear velocity corresponding to the circular frequency ω is $v \lesssim A\omega$. Using then the numerical values (2.17a) and (2.17c) we see that $v \lesssim 0.3 \times 10^{10}$ cm sec^{-1}, i.e., $v/c \lesssim 10^{-1}$, as we claimed

[7] Another possible hadronic length scale is obtained from the universal slope α' of Regge trajectories as $A \sim (\alpha')^{1/2}\hbar c \approx 0.2 \times 10^{-13}$ cm.

[8] We note that from (2.12) we get, with the value of Λ as given by (2.17b) and with a superheavy mass $\mu \sim 10^{16}$ GeV, the value $\sigma^2 \gtrsim 10^{49}$ erg^{-1} cm^{-3} for the "measure of symmetry breaking." [For comparison: also in the Salam–Weinberg theory, according to (2.15) we have $\sigma^2 \sim 0.5 \times 10^{49}$ erg^{-1} cm^{-3}.] Conversely, we may say that, if in the *weak-electromagnetic sector* the symmetry breaking is of the "usual" order of magnitude and if the characteristic "radius" $A \sim 10^{-13}$ cm, then the theory must contain a superheavy mass $\mu \sim 10^{16}$ GeV.

[9] We assume that, at least for the lowest excitations, interactions between these representative quasiparticles can be neglected, so that they are "free."

Now, it was first shown in Bacry and Lévy-Leblond (1968) that the speed-space contraction (i.e., the low velocity, small spatial distance approximation) of the de Sitter group is the Hooke group.[10] Thus, in our microuniverse model of hadrons, *the symmetries of lowly excited states will be determined by the Hooke group.*

It is well known (Bacry and Lévy-Leblond, 1968; Roman and Haavisto, 1976) that the energy of a freely moving particle with Hooke symmetry is given by the Hamiltonian operator[11, 12]

$$H = \mathbf{P}^2/2M + (M\omega^2\,\Xi^2/2) \tag{3.1}$$

which has the eigenvalues

$$E = \hbar\omega(N + 3/2), \qquad N = 0, 1, \ldots \tag{3.2}$$

Thus, in our model, these quasiparticle energy levels will correspond to the low-lying hadron states, with *one* quasiparticle present.

We can now estimate the lowest-lying hadron state energies. For N of order 1, $E \sim \hbar\omega$ and with (2.17c) we get $E \sim 10^2$ MeV, which is indeed of the order of the masses of the lightest mesons, as one would expect. A more detailed analysis will be given in Section 4. It is interesting that we can turn the argument around and thereby get a "consistency" check of our model. If we put, for the lowest excitations ($N \sim 1$), $E \sim \hbar\omega \sim 200$ MeV, we have $\omega \sim 10^{22}$–10^{23} sec^{-1}, just as we found from the microuniverse model built on the Higgs mechanism.

The hadronic microuniverses ought to be looked upon as "bubbles" imbedded in the locally flat external space-time (cf. footnote 5 above). Accordingly, the hadronic wave functions will be bilocal; one coordinate, x_μ, describing the position of the bubble in Minkowski space-time, and the other coordinate, ξ, describing the location of the quasiparticles in the internal curved space-time. Since we are interested only in the low-energy region where the description of the (low-speed) quasiparticles is approximated by the nonrelativistic Hooke group, each quasiparticle needs a three-vector $\boldsymbol{\xi}$ to be localized; hence, if we have a state with n quasiparticles, the generic symbol ξ refers to a set of $3n$ numbers.

If we have a hadronic excitation that corresponds to n (noninteracting) Hooke quasiparticles, the Hamiltonian of this system is

$$H = \sum_{i=1}^{n} \left[(\mathbf{P}^{(i)})^2/2M_i + (M_i\omega^2/2)(\boldsymbol{\xi}^{(i)})^2\right] + U \tag{3.3}$$

[10] For a derivation of the Hooke group based on local gauge groups in curved space see Roman and Haavisto (1976). Further references to the Hooke group can also be found in this paper.

[11] Since our model world has $SO(3, 2)$ rather than $SO(4, 1)$ symmetry, the relevant Hooke group is $\mathscr{H}^+$, and we use, in the following, the corresponding formulas. Cf. Bacry and Lévy-Leblond (1968) and Roman and Haavisto (1976).

[12] The frequency ω occurs in the Hooke algebra via the commutator $[H, P_k] = iM\omega^2\Xi_k$ and is the same as the oscillation frequency of the spatial sections of the de Sitter world, cf. equation (2.7); i.e., in our case, $\omega = (|\Lambda|/3)^{1/2}$. The M is one of the Casimir invariants; the position operator $\boldsymbol{\Xi}$ is related to the generator of boosts $\mathbf{Q}$ by $\boldsymbol{\Xi} = \mathbf{Q}/M$. In our model M is the quasiparticle mass.

where M_i is the mass of the ith Hooke particle and U the internal energy of the system.[13] According to our model, *we naturally identify the eigenvalue* m_0 *of (3.3) with the mass of the hadronic state corresponding to the quasi-particle system.* If the corresponding eigenstate of H is denoted by $\phi(\xi)$, this wave function is unaffected by external (restricted Poincaré) transformations. It then follows that (in our approximation) the group structure of the hadron bubbles is simply the direct product of the Poincaré and the Hooke groups, so *the bilocal wave function factorizes*

$$\Psi(x, \xi) = \psi(x)\phi(\xi) \tag{3.4}$$

In particular, under restricted Poincaré transformations we have the transformation law

$$\begin{aligned} [U(\Lambda)\Psi](x, \xi) &= \psi(\Lambda^{-1}x)\phi(\xi) \\ [U(a)\Psi](x, \xi) &= \psi(x - a)\phi(\xi) \end{aligned} \tag{3.5}$$

While the internal $\phi(\xi)$ part of the bilocal wave function is unaffected by restricted Poincaré transformations, this may not be the case for discrete transformations. For indeed we know that various single-hadron states can be assigned *intrinsic* parity and charge-parity quantum numbers. In order to determine the effect of P, C, and T symmetry transformations on the Hooke part $\phi(\xi)$ of the hadronic wave function, we examine equivalently the effect of these operations on the Hooke group generators.

It has been shown by Rosen (1965) that using the five-dimensional imbedding of the de Sitter group, one can systematically define coordinate reflections T, P, C which, in the flat-space limit, are symmetries analogous to these operations acting in Minkowski space. The action of T, P, C on the embedding space is given by

$$\begin{aligned} T&: (x_0, \mathbf{x}, x_4) \to (-x_0, \ \mathbf{x}, \ x_4) \\ P&: (x_0, \mathbf{x}, x_4) \to (\ x_0, -\mathbf{x}, \ x_4) \\ C&: (x_0, \mathbf{x}, x_4) \to (\ x_0, \ \mathbf{x}, -x_4) \end{aligned} \tag{3.6}$$

We look for the corresponding transformations when we contract to the Hooke group. Using the standard realizations[14] of the generators P_k, Q_k and remembering the antiunitarity of T, we directly find that

$$\begin{aligned} P&: Q_k \to -Q_k \\ P&: P_k \to -P_k \end{aligned} \tag{3.7a}$$

[13] We remind the reader that an irreducible unitary representation of the Hooke group is labeled by mass M, spin s, and an interal energy U. For a single particle, U can be taken to be zero; but for a system of particles (direct product of irreducible representations) this cannot be done arbitrarily for each term in the reduction.

[14] As shown in Roman and Haavisto (1976) and Bacry and Lévy-Leblond (1968)

$$Q_k \sim M\xi_k \cos \omega\tau - (i/\omega)(\sin \omega\tau)\partial_k$$
$$P_k \sim -i(\cos \omega\tau)\partial_k - M\xi_k \omega \sin \omega\tau$$

Here ξ_k corresponds to the imbedding coordinate x_k, and the historical time τ to the x_0.

whereas

$$T\colon Q_k \to +Q_k$$
$$T\colon P_k \to -P_k \tag{3.7b}$$

To see the action of C on the Hooke generators, we must follow the de Sitter generators through the contraction process. Writing the de Sitter algebra as $[M_{ab}, M_{cd}] = i(g_{ac}M_{bd} + g_{bd}M_{ac} - g_{ad}N_{bc} - g_{bc}M_{ad})$ and looking at (3.6) we see that

$$C\colon M_{ab} \to +M_{ab} \qquad \text{if } a, b \neq 4$$
$$C\colon M_{ab} \to -M_{ab} \qquad \text{if } a = 4 \text{ or } b = 4 \tag{3.8}$$

As is well known (Bacry and Lévy-Leblond) the contraction is exhibited by defining[15]

$$Q_k \equiv \lim_{\epsilon\to 0} \epsilon M_{k0} \tag{3.9a}$$

$$P_k \equiv \lim_{\epsilon\to 0} \epsilon M_{k4} \tag{3.9b}$$

$$H \equiv M_{04} \tag{3.9c}$$

$$J \equiv M_{kl} \tag{3.9d}$$

$$M \equiv \lim_{\epsilon\to 0} \epsilon^2 H \tag{3.9e}$$

In view of (3.8) we then get[16]

$$C\colon Q_k \to +Q_k$$
$$C\colon P_k \to -P_k$$
$$C\colon J_k \to +J_k \tag{3.10}$$
$$C\colon H \to -H$$

Finally, (3.9e) tells us that the last line of (3.10) implies

$$C\colon M \to -M \tag{3.11}$$

These results are, naturally, also consistent with the commutators $[P_k, Q_l] = -iM\delta_{kl}$ etc. of the Lie algebra. We note that, since the position operator $\boldsymbol{\Xi} \equiv \mathbf{Q}/M$, equations (3.10) and (3.11) tell us that

$$C\colon \Xi_k \to -\Xi_k \tag{3.12}$$

[15] We consider the generators M_{ab} as functions of $\epsilon \sim A^{-1}$, where A is the amplitude of the de Sitter world.

[16] At this point we mention that, because of (3.7a), (3.7b), and (3.1), under P and T neither H nor [in view of (3.9e)] M changes. From consistency of the Lie algebra it follows that J_k does not change under P but changes sign under T.

Next, we observe that $H \to -H$ implies $E \to -E$. In view of (3.2) we therefore have, for the system with H, the energy spectrum of a single Hooke particle

$$E = (N + 3/2)\hbar\omega, \qquad N \geqslant 0 \tag{3.13a}$$

but for the C-transformed system with $-H$ we must write

$$E' = (-N - 3/2)\hbar\omega, \qquad N \geqslant 0 \tag{3.13b}$$

The spectrum of the original system starts at the ground state with $E_0 = \frac{3}{2}\hbar\omega$ and goes upward, whereas the C-transformed system has its ground state at $E'_0 = -\frac{3}{2}\hbar\omega$ and then goes downward. We can express these results by saying that

$$C\colon N \to -N \tag{3.14}$$

In other words, C-transformed Hooke particles are characterized by *negative energy and negative energy level occupation operator eigenvalues.* In terms of raising–lowering operators, therefore, the C-transformed Hooke level-raising operator (which makes $N \to N + 1$) annihilates the ground state, whereas the lowering operator increases the *magnitude* of energy by one unit $\hbar\omega$.

Finally we note that because of the realization $H - i\partial_\tau$ and because of (3.10), the historical time parameter τ changes sign under C transformations.[17]

We conclude this section with an important observation. In our model we identify the energy of excited states with the mass of the hadron. Since C-transformed Hooke particles (and thus, collective excitations that correspond to states made up from such quasiparticles) have, formally, in the non-second-quantized description negative energies, whereas we must have $m > 0$, we have a meaningful model only if we interpret (when allowance is made for second quantization) these negative-energy Hooke particles in the framework of a "hole theory." That is, *a negative-energy Hooke particle will have to be considered as an antiquasiparticle with positive energy, i.e.,*

$$E_{\text{anti-q.p.}} = |E'| = (N + 3/2)\hbar\omega \tag{3.15}$$

Since a hole theory makes sense only if the "particles" obey Fermi statistics, we must accept the fact that the Hooke particles are *fermions.* In a nonrelativistic theory the spin-statistics theorem cannot be proven and it plays the role of a postulate. Thus we are led to conclude that the Hooke representations (which describe the quasiparticles that are brought about by the unknown dynamics of the system) belong to half-integral spin. For simplicity we then take all emerging *quasiparticles* (i.e., Hooke representations) *to have* $s = 1/2$.

4. *SU*(3) *Structure and Connection with the Quark Model*

At this point, we can make contact with the simple quark model. We saw that the energy operator corresponding to an n-quasiparticle excitation of the Hooke model is given by equation (3.3). On the other hand, the starting point

[17] This is also consistent with the Q_k, P_k realizations given in footnote 14. Observe that, because of (3.12), we have $\xi_k \to -\xi_k$.

for the standard naive nonrelativistic quark model (Faiman and Hendry, 1968) is the Hamiltonian

$$H = \sum_{i=1}^{n} (\mathbf{P}^{(i)})^2/2M + (M\omega^2/2) \sum_{i<j}^{n} (r^{(i)} - r^{(j)})^2 \tag{4.1}$$

where M is the average of the M_i. This expression is then transformed to center-of-mass coordinates, and the CM motion is ignored as unphysical ("spurious" states). The resulting Hamiltonian corresponds to that of the shell model:

$$H = \sum_{i=1}^{n} [(\mathbf{P}^{(i)})^2/2m_i + m\omega^2 (r^{(i)})^2] + E^C \tag{4.2}$$

where now the coordinates refer to quark–CM distances, m_i are the reduced masses (with average m), and E^C is the CM kinetic energy. Comparing (4.2) with (3.3) we thus see that if our quasiparticles can be given quark [i.e., $SU(3)$] quantum numbers,[18] then *the Hooke description of our model is equivalent to a nonrelativistic quark model with harmonic force interactions.* One immediate advantage of this viewpoint is that no quark confinement problem arises: The "quarks" are simply *quasi*particles that are used to characterize the collective excitations of our model. Below we pursue these ideas in some detail.

4.1. Quasiparticle Spectrum and SU(3). The one-quasiparticle excitation energy spectrum of our model is given by equation (3.1). To exhibit the well-known $SU(3)$ degeneracy of the energy levels, we introduce the standard level-raising and -lowering operators,

$$a_k = (2^{-1/2}) [(M\omega/\hbar)^{1/2}\Xi_k - i(\hbar/M\omega)^{1/2}P_k] \tag{4.3a}$$

$$a_k^\dagger = (2^{-1/2}) [(M\omega/\hbar)^{1/2}\Xi_k + i(\hbar/M\omega)^{1/2}P_k] \tag{4.3b}$$

and define the operators $A_{kl} \equiv a_k^\dagger a_l$ which are known to obey the $U(3)$ algebra. It is convenient to introduce the linear combinations

$$\begin{aligned} I_+ &= a_1^\dagger a_2, & I_- &= a_1 a_2^\dagger, & I_3 &= \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \\ U_+ &= a_2^\dagger a_3, & U_- &= a_2 a_3^\dagger, & U_3 &= \tfrac{1}{2}(a_2^\dagger a_2 - a_3^\dagger a_3) \\ V_+ &= a_1^\dagger a_3, & V_- &= a_1 a_3^\dagger, & V_3 &= \tfrac{1}{2}(a_1^\dagger a_1 - a_3^\dagger a_3) \end{aligned} \tag{4.4}$$

which are the familiar generators of $SU(3)$, and also

$$N = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 \tag{4.5}$$

which generates $U(1)$. The usual diagonal operators used for labeling states can then be expressed as

$$\begin{aligned} Q &\equiv \tfrac{2}{3}(I_3 + 2V_3) = \tfrac{2}{3}a_1^\dagger a_1 - \tfrac{1}{3}a_2^\dagger a_2 - \tfrac{1}{3}a_3^\dagger a_3 \\ Y &\equiv \tfrac{2}{3}(2U_3 + I_3) = \tfrac{1}{3}(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3) \\ I_3 &= \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \end{aligned} \tag{4.6}$$

[18] We already know that the spin quantum number of our quasiparticles is 1/2.

If we denote the eigenvalues of $a_k^\dagger a_k$ by n_k (which is an integer) then we can write

$$Q = \tfrac{2}{3}n_1 - \tfrac{1}{3}n_2 - \tfrac{1}{3}n_3 \tag{4.7a}$$

$$Y = \tfrac{1}{3}(n_1 + n_2 - 2n_3) \tag{4.7b}$$

$$I_3 = \tfrac{1}{2}(n_1 - n_2) \tag{4.7c}$$

For any fixed value of E, that is, for any given energy level determined by the eigenvalue of N, the degenerate eigenstates span an irreducible representation of $SU(3)$, whose dimension equals the degree of degeneracy. In particular, for $N = 1$, the state vectors have the form $|1\rangle = |n_1\rangle\,|n_2\rangle\,|n_3\rangle$, where n_k is the occupation number for oscillation in the ξ_k direction and $n_1 + n_2 + n_3 = 1$. We have then the basis

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \equiv |1\rangle\,|0\rangle\,|0\rangle, \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \equiv |0\rangle\,|1\rangle|0\rangle, \qquad \begin{pmatrix}0\\0\\1\end{pmatrix} \equiv |0\rangle\,|0\rangle\,|1\rangle \tag{4.8}$$

which spans the representation $\{\mathbf{3}\}$. If we consider, instead of a Hooke particle, the corresponding C-conjugate particle, then, as was discussed in Section 3, N changes sign and thus the n_k are negative. The basis $|-1\rangle\,|0\rangle\,|0\rangle$ etc. carries the representation $\{\mathbf{3^*}\}$, which can be seen from the fact that, as (4.7) shows, the diagonal operators have eigenvalues of opposite sign, We should, however, remember that, as discussed at the end of Section 3, the physical antiquasiparticle energy associated with the $\{\mathbf{3^*}\}$ representation (as well as of the entire antiquasiparticle spectrum) is positive.

Using the basis (4.8) and substituting the appropriate n_k into (4.7) we see that *the first excitation level of the Hooke quasiparticle carries precisely the well-known SU(3) quark quantum numbers.* Similarly, the first level of the antiquasiparticle is associated with the antiquark quantum numbers. This, then, establishes the desired link.

Even though, as noted above, the excitation spectrum of a single quasiparticle/antiquasiparticle reproduces all irreducible representations of $SU(3)$, in the following we will not be interested in these higher levels of single excitation; rather, we wish to study systems in which there are several quasiparticles (and/or antiquasiparticles), with *each of them occupying the level* $N = 1$. (The physical reason for this restriction will be discussed shortly.) Furthermore, in our first approximation for the low-lying hadron states we shall assume (as mentioned earlier) that the quasiparticles do not interact, i.e., they are free Hooke particles. In consequence of these assumptions, the multiquasiparticle states we consider will be direct products of $\{\mathbf{3}\}$ and/or $\{\mathbf{3^*}\}$ representations of $SU(3)$, and will span, in general, *reducible* representations.

In order to discuss these multiexcitation states in terms of Hooke generators, we define level-raising and -lowering operators

$$^{(i)}a_k^\dagger,\ ^{(i)}a_k \qquad (i = 1, 2, \ldots, n;\ k = 1, 2, 3)$$

for the n distinct quasiparticles.[19] To consider a specific example, let us take a three-quasiparticle system. From equation (3.3) it then follows[20] that, in general,

$$E_\alpha = (N^{(1)} + N^{(2)} + N^{(3)} + \tfrac{9}{2})\hbar\omega + U_\alpha \qquad (\alpha = 1, \ldots, r) \tag{4.9}$$

Here

$$N^{(i)} = \sum_{k=1}^{3} {}^{(i)}a_k^\dagger \, {}^{(i)}a_k \qquad (i = 1, 2, 3) \tag{4.10}$$

and these operators have the eigenvalues

$$n^{(i)} = \sum_{k=1}^{3} n_k^{(i)} \qquad (i = 1, 2, 3) \tag{4.11}$$

where each $n_k^{(i)}$ is an integer. The U_α Hooke internal energies are in general different for the r irreducible components $\alpha = 1, 2, \ldots, r$ that occur in the product representation. *In keeping with our low-energy (low-speed) approximation, we must take all three excitation levels to be the lowest,*[21] i.e., we take

$$n^{(1)} = n^{(2)} = n^{(3)} = 1 \tag{4.12}$$

Thus, each of the three quasiparticles is occupying its first level of excitation, so that, according to the preceding analysis, it belongs to the $\{3\}$ representation of $SU(3)$. Consequently, our three-quasiparticle system carries the product representation

$$\{3\} \times \{3\} \times \{3\} = \{1\} + \{8\} + \{8\} + \{10\} \tag{4.13}$$

With (4.12) and (4.11) we then obtain from the general formula (4.9) the energy spectrum for this supermultiplet:

$$E_\alpha = \tfrac{15}{2}\hbar\omega + U_\alpha \qquad (\alpha = 1, \ldots, 4) \tag{4.14}$$

for the masses of the 27 possible hadron states contained in (4.13). The mass splitting of the four irreducible components is accounted for by the Hooke internal energies U_α, which cannot be calculated since the dynamics that gives rise to the Hooke quasiparticles is not encompassed by our model. Within each irreducible component we have mass degeneracy.

In general, for an n-quasiparticle excitation, the Casimir invariants depend not only on

$$N \equiv \sum_{i=1}^{n} n^{(i)} \tag{4.15}$$

[19] For antiquasiparticles, the remark following equation (3.14) must be remembered.

[20] Since we disregard interactions, and since the (unknown) dynamical mechanism that brings about the quasiparticles cannot distinguish them, the masses M_i ($i = 1, 2, 3$) will be the same.

[21] If we permitted any $n^{(i)} > 1$, the expectation value of $\mathbf{P}^{(i)}$ may become too large for the Hooke group to be applicable. Note that, as discussed in Section 2, even for a single quasiparticle and $N = 1$, we have $v/c \sim 0.1$, i.e., the validity of nonrelativistic motion is marginal.

but also on the total number of quasiparticles "oscillating" in each direction, i.e., on the quantities

$$n_k \equiv \sum_{i=1}^{n} n_k^{(i)} \qquad (k = 1, 2, 3) \tag{4.16}$$

If N is fixed, the different irreducible representations can be characterized by identifying the maximum number of quasiparticles "oscillating" in any one of the three directions. For example, in the specific case studied above, $N = 3$ and we find (by constructing the possible product eigenfunctions belonging to the degenerate oscillator states) that

$$\max n_k = \begin{cases} 1 \text{ for } \{\mathbf{1}\} \\ 2 \text{ for } \{\mathbf{8}\} \\ 3 \text{ for } \{\mathbf{10}\} \end{cases} \tag{4.17}$$

Another interesting example is the study of an excited state that contains one quasiparticle and one antiquasiparticle. Then, in general, we have

$$E_\alpha = (N^{(1)} + \bar{N}^{(2)} + 3)\hbar\omega + U_\alpha \qquad (\alpha = 1, 2, \ldots, r) \tag{4.18}$$

Once again, in the spirit of our low-energy approximation, we must restrict ourselves to the values

$$n^{(1)} = \bar{n}^{(2)} = 1$$

so that

$$E_\alpha = 5\hbar\omega + U_\alpha \qquad (\alpha = 1, 2) \tag{4.19}$$

and the nine states span the product representation

$$\{\mathbf{3}\} \times \{\mathbf{3}^*\} = \{\mathbf{1}\} + \{\mathbf{8}\}$$

The splitting between the singlet and octet mass is determined by the U_α. We now have $N = 2$ fixed and, writing $n_k \equiv n_k^{(1)} + \bar{n}_k^{(2)}$, we find that

$$\max n_k = \begin{cases} 1 \text{ for } \{\mathbf{1}\} \\ 2 \text{ for } \{\mathbf{8}\} \end{cases}$$

in conformity with (4.17).

4.2. Quasiparticles and Quarks. In order to discuss in more detail the decomposition of the product representations that are associated with multiple quasiparticle systems, each in its lowest state ($n^{(i)} = 1$) of excitation, we must perform the reduction of the product with respect to a specific subgroup chain. The most illuminating approach to do this is to fully utilize the connection between our quasiparticle description and the standard $SU(3)$ quark model, and to use the $SU(3) \supset SU(2) \times U(1)$ chain. We already know that the Hooke quasiparticles have spin $\frac{1}{2}$ and their $n = 1$ level states have quark $SU(3)$ quantum numbers. In particular, from (4.8) and (4.7) we see that the u, d, s quarks

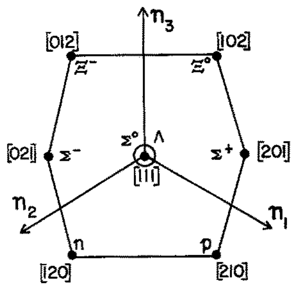

Figure 1. Baryon octet.

correspond to an $N = 1$ quasiparticle that "oscillates" in the ξ_1, ξ_2, ξ_3 direction, respectively. We shall write generically for a system in which the *total* number of occupied oscillation modes in the ξ_1, ξ_2, ξ_3 directions is n_1, n_2, n_3, respectively, the symbol

$$A \equiv [n_1 n_2 n_3] \tag{4.20}$$

Thus, in particular,

$$u = [100], \qquad d = [101], \qquad s = [001]$$

Similarly, for the antiquarks[22] we have

$$\bar{u} = [-100], \qquad \bar{d} = [0-10], \qquad \bar{s} = [00-1]$$

It is now evident that, if we consider systems with several quasiparticle and/or antiquasiparticle excitations present (each in its $n^{(i)} = n_1^{(i)} + n_2^{(i)} + n_3^{(i)} = \pm 1$ level), the sum of the entries in the generic "particle symbol" (4.20) will give the number of quarks minus the number of antiquarks. Thus, in any (degenerate) multi-quasiparticle/antiquasiparticle system (with each $n^{(i)} = \pm 1$) we have, for all members of all irreducible $SU(3)$ multiplets that are contained in the reducible representation, the *same* value for $n_1 + n_2 + n_3$ = No. of quarks – No. of antiquarks. Moreover, the entries n_1, n_2, n_3 in the symbol (4.20) will tell us how many *u, d, s* quarks and/or $\bar{u}$, $\bar{d}$, $\bar{s}$ antiquarks are present.

For example, considering again the case when we have three quasiparticles (each in its first excited state), $n_1 + n_2 + n_3 = 3$. Because of the additivity of the level occupation numbers, the isospin component and hypercharge quantum numbers of these states can be correlated to the symbol (4.20) by using equations (4.7). As an illustration, Figure 1 shows the baryon octet, conveniently displayed in a triangular coordinate system, with axes n_1, n_2, n_3 and centered about the point [111]. The point [102], for example, designates the system *uss*, i.e., the Ξ^0 particle.

[22] Note that, because of the hole-theory picture, the antiquasiparticle sublevel occupation numbers $\bar{n}_k$ are the negatives of the *C*-conjugate Hooke particle sublevel occupation numbers n_k.

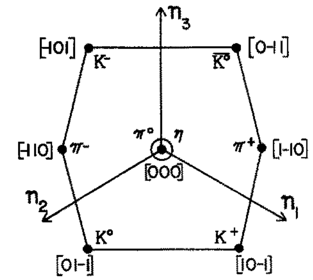

Figure 2. Meson octet.

The meson octet (which is part of the reducible representation composed of one quasiparticle and one antiquasiparticle—each of them in the first excited level) is illustrated in Figure 2. In this case $n_1 + n_2 \equiv n_1 - \bar{n}_2 = 0$. The quark composition is again evident from the label $[n_1 n_2 n_3]$ of the points.

In general, given n_1, n_2, n_3, there can be several different particles $[n_1 n_2 n_3]$ which belong to different $SU(3)$ multiplets and/or $SU(2)$ submultiplets. As in the quark model, these questions can be best described by considering the symmetries of the complete wave function. Since the Hooke quasiparticles have spin $\frac{1}{2}$, the energy eigenstates will have an over-all $SU(6) \supset SU(3) \times SU(2)$ $SU(2)_{\text{spin}}$ structure, and we must distinguish identically labeled $[n_1 n_2 n_3]$ states in relation to their symmetry under $SU(6)$. Consider, for example, the three-quasiparticle (baryon) states. With respect to spin we may have fully symmetric states (they have $s = \frac{3}{2}$) and mixed symmetry states (they have $s = \frac{1}{2}$). Further, the physical $SU(3)$ states are fully symmetric (they belong to $\{\mathbf{10}\}$) or have mixed symmetry (these belong to $\{8\}$). To be specific, consider the (positive parity) baryons that are characterized by our symbol $A = [111]$, i.e., the excitations with three quasiparticles where all three are on the first energy level ($n^{(1)} = n^{(2)} = n^{(3)} = 1$) and where one quasiparticle "oscillates" in each direction. There will be three such states. One will be symmetric with respect to spin ($s = \frac{3}{2}$) and symmetric with respect to $SU(3)$ (i.e., belongs to $\{\mathbf{10}\}$). This is the $I_3 = 0$ member of the Y^* (1385) isotriplet. The other two $[111]$ states have mixed spin symmetry (i.e., $s = \frac{1}{2}$) and mixed $SU(3)$ symmetry (belong to $\{8\}$). In the $SU(3) \supset SU(2) \times U(1)$ decomposition they can be distinguished as the Σ^0 and the Λ, respectively.

4.3. Relative Phases between Quasiparticles. We can gain further insight into the nature of multi-quasiparticle/antiquasiparticle systems if we adopt a suitable maximal set of commuting observables for labeling the states. Generalizing the work of Carruthers and Nieto (1968) we shall construct observables that measure averaged relative phases between the oscillators that correspond to our quasiparticles.

Let us introduce, for a single given Hooke oscillator, the "renormalized" level-lowering and -raising operators

$$A_k \equiv (N+1)^{-1/2} a_k, \qquad A_k^\dagger \equiv a_k^\dagger (N+1)^{-1/2} \qquad (k = 1, 2, 3) \tag{4.21}$$

where $N = (N_1 + N_2 + N_3)$ and which act on the states as follows:

$$A_k |0\rangle = 0, \qquad A_k |n_k\rangle = |n_k - 1\rangle, \qquad A_k^\dagger |n_k\rangle = |n_k + 1\rangle \tag{4.22}$$

Thus A_k is a "partial isometry":

$$A_k^\dagger A_k = 1 - |0\rangle\langle 0|, \qquad AA^\dagger = 1 \tag{4.23}$$

We have the commutation relations

$$[A_k, A_l^\dagger] = |0\rangle \delta_{kl} \langle 0|, \qquad [A_k, A_l] = [A_k^\dagger, A_l^\dagger] = 0 \tag{4.24}$$

Let us now define

$$C_k \equiv \tfrac{1}{2}(A_k^\dagger + A_k), \qquad S_k \equiv \frac{1}{2i}(A_k^\dagger - A_k) \tag{4.25}$$

One easily verifies that

$$\begin{aligned} [C_k, S_l] &= \frac{1}{2i} |0\rangle \delta_{kl} \langle 0| \\ [C_k, C_l] &= [S_k, S_l] = 0 \end{aligned} \tag{4.26}$$

We also find from (4.23) that

$$C_k{}^2 + S_k{}^2 = 1 - |0\rangle \tfrac{1}{2} \langle 0| \qquad (k = 1, 2, 3) \tag{4.27}$$

Furthermore, with $H = (N_1 + N_2 + N_3 + \frac{3}{2})\hbar\omega$, we get

$$[H, C_k] = i\hbar\omega S_k, \qquad [H, S_k] = -i\hbar\omega C_k \tag{4.28}$$

Equations (4.27) and (4.28) suggest that we should introduce "trigonometric operators." Indeed, let us write

$$A_k^\dagger = e^{i\phi_k} \tag{4.29a}$$

$$A_k = e^{-i\phi_k} - e^{-i\phi_k} |0\rangle\langle 0| \tag{4.29b}$$

This can be done, since as one easily checks,[23] this angle representation satisfies the algebraic relations (4.23), (4.24). Then from (4.25), (4.29) we obtain

$$C_k = \cos\phi_k - e^{-i\phi_k} |0\rangle \frac{1}{2} \langle 0| \tag{4.30a}$$

$$S_k = \sin\phi_k + e^{-i\phi_k} |0\rangle \frac{1}{2i} \langle 0| \tag{4.30b}$$

[23] In checking these and subsequent formulas, one must realize that $|0\rangle\langle 0| e^{i\phi_k}$ annihilates all states since, by (4.29a), $e^{i\phi_k} |n_k\rangle = |n_k + 1\rangle$.

and one can again check[23] that (4.26), (4.27), (4.28) are satisfied. We note that, in our Heisenberg picture, ϕ_k depends on τ, and we write conveniently

$$\phi_k(\tau) = \omega\tau + \psi_k \tag{4.31}$$

where the operator ψ_k is τ independent. Equation (4.31) is consistent with the equations of motion (4.28).

From (4.28) and from (4.30) we obtain[24]

$$[N_k, i\phi_l] = \delta_{kl}, \qquad [\phi_k, \phi_l] = 0 \tag{4.32}$$

Thus, the map $(a_k, a_k^\dagger) \to (N_k, i\phi_k)$ is a nonlinear *canonical transformation*. Actually, it can be verified that

$$\begin{aligned} \Xi_k &= (2J_k/M\omega)^{1/2}\cos\phi_k \\ P_k &= -(2M\omega J_k)^{1/2}\sin\phi_k \end{aligned} \tag{4.33}$$

where $J_k \equiv \hbar(N_k + \frac{1}{2})$. Thus, $(\Xi_k, P_k) \to (J_k, -\phi_k)$ is also a canonical transformation, and equation (4.33) clearly displays the physical meaning of ϕ_k: This operator is indeed *the phase of the mode k* of "oscillation" of the Hooke quasiparticle.

Of course, the phase of a single oscillator is of no physical consequence, and this is reflected in the fact that, because of (4.32), H and ϕ_k do not commute. However, in a system of several oscillators, relative phases should be meaningful. Indeed, one can easily define operators that measure pairwise phase differences. For simplicity of writing, let us use the notation

$$A_k \equiv {}^{(1)}A_k, \qquad B_k \equiv {}^{(2)}A_k$$

for the A_k operators relating to two distinct quasiparticles a and b, respectively, and define

$$C_{ab} = \frac{1}{6}\sum_{k,j=1}^{3} A_k B_j^\dagger + A_k^\dagger B_j \tag{4.34a}$$

$$S_{ab} = \frac{-i}{6}\sum_{k,j=1}^{3} A_k B_j^\dagger - A_k^\dagger B_j \tag{4.34b}$$

With

$$H = \sum_{k=1}^{3}\left(a_k^\dagger a_k + b_k^\dagger b_k + \cdots + \frac{3n}{2}\right)\hbar\omega$$

being the n-quasiparticle Hamiltonian, we then find that

$$[H, C_{ab}] = [H, S_{ab}] = 0 \tag{4.35}$$

[24] Note also that $[C_k, N_l] = [S_k, N_l] = 0$ for $k \neq l$.

so that either C_{ab} or S_{ab} can be simultaneously diagonalized[25] with H, as desired. Furthermore, with (4.29), equation (4.39) can be brought to the form

$$C_{ab} = \frac{1}{3} \sum_{k,j=1}^{3} \left(\cos \Delta_{kj} - e^{i\Delta_{kj}} |0\rangle^a \frac{1}{2}\, {}^a\langle 0| - e^{-i\Delta_{kj}} |0\rangle^b \frac{1}{2}\, {}^b\langle 0| \right) \tag{4.36a}$$

$$S_{ab} = \frac{1}{3} \sum_{k,j=1}^{3} \left(\sin \Delta_{kj} - e^{i\Delta_{kj}} |0\rangle^a \frac{1}{2i}\, {}^a\langle 0| + e^{-i\Delta_{kj}} |0\rangle^b \frac{1}{2i}\, {}^b\langle 0| \right) \tag{4.36b}$$

where

$$\Delta_{kj} \equiv \phi_k{}^b(\tau) - \phi_j{}^a(\tau) = \psi_k{}^b - \psi_j{}^a \tag{4.37}$$

is independent of τ, in accord with (4.35).

It may be useful to define

$$E^+_{ab} \equiv C_{ab} + iS_{ab} = \frac{1}{3} \sum_{k,j=1}^{3} A_k^\dagger B_j \tag{4.38a}$$

$$E^-_{ab} \equiv C_{ab} - iS_{ab} = \frac{1}{3} \sum_{k,j=1}^{3} A_k B_j^\dagger \tag{4.38b}$$

With (4.36) these can be expressed in the form

$$E^+_{ab} = \frac{1}{3} \sum_{k,j=1}^{3} \exp(i\Delta_{kj})(1 - |0\rangle^a\, {}^a\langle 0|) \tag{4.39a}$$

$$E^-_{ab} = \frac{1}{3} \sum_{k,j=1}^{3} \exp(-i\Delta_{kj})(1 - |0\rangle^b\, {}^b\langle 0|) \tag{4.39b}$$

Thus, $E^\pm_{ab}$ *measure suitably averaged phase differences.* Because of (4.35)

$$[H, E^\pm_{ab}] = 0 \tag{4.40}$$

Thus, we can use the phase difference (as determined, say,[26] by E^+_{ab}) as a good quantum number, along with H.

We will use this circumstance to demonstrate that *hadronic excitations which, in the quark language, would correspond to states with nonzero triality* (i.e., to states with, say, one quark, or with two quarks) *cannot arise.*[27] The argument rests on the fact that such states would *violate the Pauli symmetry*

[25] However, C_{ab} and S_{ab} do not commute.

[26] Note that E^+_{ab} and E^-_{ab} do not commute.

[27] We already mentioned that "quarks are confined" since in our model the quasiparticles are not constituents but simply a description of collective excitation modes. However, we so far left the question open as to whether collective excitations with quantum numbers of a single quark or of a di-quark, etc., can or cannot arise.

of systems built from identical particles. Indeed, equation (4.39a) shows that, because the action of E^+_{ab} is privileged with respect to the quasiparticle a, we cannot, in general, expect E^+_{ab} eigenstates that have a definite symmetry against interchange of a and b.

We illustrate the situation for the case when the excitation is supposed to have the quantum numbers of a single quark. This means the system would contain, on the first excited level, one Hooke particle. All other Hooke oscillators are in the ground state. Let us compare the phase of the excited oscillator to any one of the oscillators in the ground state. The general form of the relevant part of the state vector is

$$|\Psi\rangle = \sum_{l=1}^{3} (\alpha_l |1_l\rangle^a |0\rangle^b + \alpha^l |0\rangle^a |1_l\rangle^b) \tag{4.41}$$

where $|1_l\rangle$ is a state with the l-direction vibration mode excited, i.e.,

$$|1_1\rangle = |1\rangle|0\rangle|0\rangle, \qquad |1_2\rangle = |0\rangle|1\rangle|0\rangle, \qquad |1_3\rangle = |0\rangle|0\rangle|1\rangle$$

Applying (4.38a) we get

$$E^+_{ab}|\Psi\rangle = \frac{1}{3}(\alpha^1 + \alpha^2 + \alpha^3) \sum_{l=1}^{3} |1_l\rangle^a |0\rangle^b \tag{4.42}$$

If $|\Psi\rangle$ is an eigenstate with eigenvalue e, then we have $E^+_{ab}|\Psi\rangle = e|\Psi\rangle$, so that (4.41) and (4.42) give the conditions

$$\alpha^l = 0 \qquad (l = 1, 2, 3)$$

and

$$3e\alpha_k = \sum_{l=1}^{3} \alpha^l = 0 \qquad (k = 1, 2, 3)$$

The second condition means that either $\alpha_k = 0$ ($k = 1, 2, 3$) so that $|\Psi\rangle$ is the null vector; or $e = 0$, whence the α_k are arbitrary so that the $e = 0$ eigenstate would be

$$|\Psi\rangle = \sum_{l=1}^{3} \alpha_l |1_l\rangle^a |0\rangle^b$$

But this is neither symmetric nor antisymmetric against interchange of a and b, so that the Pauli principle is violated.

Another illustration is the case of a "two-quark" excitation, i.e., where we consider an energy eigenstate with $N = N^{(a)} + N^{(b)} = 2$, and ask whether such states, being simultaneous eigenstates of H and E^+_{ab}, can consist of both oscillators on the first level and having proper Pauli (anti)symmetry against exchange. The general state vector must now be written in the form

$$|\Psi\rangle = \sum_{k,l=1}^{3} (\alpha^{kl}|1_k 1_l\rangle^a |0\rangle^b + \alpha_l{}^k |1_k\rangle^a |1_l\rangle^b + \alpha_k{}^l |1_l\rangle^a |1_k\rangle^b + \alpha_{kl}|0\rangle^a |1_k 1_l\rangle^b)$$

One verifies (with a somewhat lengthy calculation[28]) that it is impossible to have E^+_{ab} eigenstates with any definite symmetry.

Of course, if one considers quasiparticle-antiquasiparticle states (i.e., quark-antiquark systems) then the Pauli principle is not operative so that there is no obstacle to having such states appear as acceptable simultaneous eigenstates of H and E^+_{ab}.

5. *Summary*

We have demonstrated that in completely unified spontaneously broken gauge theories, where superheavy bosons are expected to be present, the vacuum expectation value of the Hamiltonian (and hence, the value of the corresponding cosmological constant) may be so large as to cause an enormous curvature in the small with the characteristic scale factor of the spacelike sections having the order of the typical extension of a hadron. Presence of "ordinary" subnuclear matter has negligible effect. In particular, if the symmetry breaking is caused (or described) by a Higgs-type mechanism, the cosmological constant is negative and the solution of the Einstein equations has $SO(3, 2)$ symmetry. In the low-excitation (low-speed) limit the dynamics is then described by the Hooke group, to which corresponds an oscillator-like Hamiltonian. The collective excitations can then be characterized in terms of quasiparticles and antiquasiparticles, which are Hooke oscillator modes. Multi-quasiparticle excitations carry $SU(3)$ quantum numbers, such states being direct products of the $\{\mathbf{3}\}$ and $\{\mathbf{3}^*\}$ representations, where we restrict ourselves to each Hooke oscillator being in its first excited state. (This is in keeping with our low-energy limit.) Detailed contact with the standard naive nonrelativistic quark model can be established. The "quarks" (being simply *quasi*particles) are naturally confined and, because eigenstates of a suitably constructed relative phase operator cannot possess Pauli exchange symmetry, collective excitations with nonzero triality are not permitted.

We find it intriguing that the $SU(3)$ structure is contingent upon the fact that the Hooke group is the low-speed (*low-energy excitation*) limit of the symmetry group $SO(3, 2)$ associated with the solution of the Einstein equations. It is not difficult to see that there exists a "relativistic Hooke group," which arises as the contraction, not of $SO(3, 2)$ but of $SO(3,3)$. One wonders whether "microuniverse solutions" exist for spontaneously broken unified gauge theories that have, to start with, this higher symmetry and whether, after a suitable limit is taken, the quasiparticle systems describing the collective excitations would exhibit an $SU(4)$ structure, as is nowadays expected for more energetic excitations of subnuclear matter. We shall investigate this possibility in the future.

It should be obvious that, in any case, our present work must not be considered as a well-formed theory: It only suggests a possible and, it appears, consistent, speculative framework. Improvements and refinements (even apart

[28] One must take into account that, by isotropy $|\alpha^{kl}|^2 = |\alpha^{lk}|^2$, $|\alpha_l{}^k|^2 = |\alpha_k{}^l|^2$, $|\alpha^{kl}|^2 = |\alpha_{kl}|^2$. The normalization conditions give further constraints.

from the possible extension to charm as suggested in the preceding paragraph) are necessary and do not appear impossible. Surely one ought to study the response of the quasiparticles ("quarks") to external probing by fields—this study should then show how deep inelastic scattering behavior and/or asymptotic freedom arises. Likewise, we have, at this time, no clue to explain large transverse momentum effects. Finally, symmetry breaking and hence, removal of mass degeneracies in the multiplets must result from interactions between the quasiparticle excitations, which, presently, we do not know how to introduce in a unique manner. These and many other questions deserve further study.

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